

COMPLETELY SEMI- φ -MAPS

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ABSTRACT. We introduce completely semi- φ -maps on Hilbert C^* -modules as a generalization of φ -maps. This class of maps provides examples of CP-extendable maps which are not CP-H-extendable, in Skeide-Sumesh's sense. Using the CP-extendability of completely semi- φ -maps, we give a representation theorem, similar to Stinespring's representation theorem, for this class of maps which can be considered as strengthened and generalized form of Asadi's and Bhat-Ramesh-Sumesh's analogues of Stinespring representation theorem for φ -maps. We also define an order relation on the set of all completely semi- φ -maps and establish a Radon-Nikodym type theorem for this class of maps in terms of their representations.

1. INTRODUCTION

Asadi in [3] and Bhat-Ramesh-Sumesh in [4] gave a representation theorem for a class of maps on Hilbert C^* -modules, as a generalization of Stinespring's representation theorem for completely positive maps on C^* -algebras. Michael Skeide [9] achieved a generalization of Bhat-Ramesh-Sumesh's theorem for φ -maps between Hilbert C^* -modules in term of C^* -correspondences.

L. Arambašić [1], extended the representation theory of C^* -algebras to Hilbert C^* -modules. She showed that the set of all representations of a Hilbert C^* -module \mathcal{E} are in one to one correspondences with the set of all representations of its linking C^* -algebra $\mathcal{L}(\mathcal{E})$. Skeide and Sumesh [10] introduced CP-extendable maps between Hilbert C^* -modules as maps that can be extended to a completely positive map acting block-wise between the associated (extended or reduced) linking algebras. They characterized φ -maps in term of those CP-extendable maps where the 11-corner of the extension can be chosen to be a homomorphism, the CP-H-extendable. Besides of the studying CPH-semigroups in [10], they presented a factorization of strictly CP-extendable maps, too. Combining Stinespring's representation theorem [11, 8] and a result of L. Arambašić [1, Proposition 3.1] imply that an operator valued map on a Hilbert C^* -module is dilatable if and only if it is CP-extendable.

The above results and facts motivate us to take a closer look at dilatable maps and provide a class of CP-extendable maps which are not CP-H-extendable. We introduce the class of completely semi- φ -maps as a generalization of φ -maps. We concentrate on operator valued completely semi- φ -maps and strengthen Skeide-Sumesh' theorem [10, Theorem 1.3] in operator valued case by showing that completely semi- φ -maps are exactly those CP-extendable maps which its 11-corner of the extension can be chosen to be a unital completely positive map. Using this we strengthen the main result of [3] for completely semi- φ -maps in Section 3.

Moreover, we use the minimality conditions that was introduced in [4] for dilation pairs of φ -maps to introduce the minimal dilation pairs of completely semi- φ -maps, and show that two minimal dilation pairs for a given completely semi- φ -maps on a Hilbert C^* -module \mathcal{E} implement unitarily equivalent $*$ -representations on the linking C^* -algebra $\mathcal{L}_1(\mathcal{E})$. Furthermore, we give two characterizations of completely semi- φ -maps in terms of their CP-extension and minimal dilation pairs which helps us to construct examples of completely semi- φ -maps.

2010 *Mathematics Subject Classification.* Primary: 46L08, Secondary: 46L07.

Key words and phrases. Hilbert C^* -modules, Stinespring's theorem, completely positive maps.

In Section 5, we define an order relation on the set of all completely semi- φ -maps and provide a Radon-Nikodym type theorem for completely semi- φ -maps in terms of their dilation pairs. This Radon-Nikodym type theorem for completely semi- φ -maps maps strengthen the Joita's result on φ -maps on Hilbert C^* -modules (c.f. [7, Theorem 2.15]).

2. PRELIMINARIES

For a right Hilbert C^* -module \mathcal{E} over a unital C^* -algebra \mathcal{A} , the linking C^* -algebra of \mathcal{E} is denoted by $\mathcal{L}(\mathcal{E})$ and defined as $\mathcal{L}(\mathcal{E}) := \left\{ \begin{bmatrix} u & x \\ y^* & a \end{bmatrix} \mid a \in \mathcal{A}, u \in \mathbb{K}(\mathcal{E}), x, y \in \mathcal{E} \right\}$, where $\mathbb{K}(\mathcal{E})$ is the set of compact operators on \mathcal{E} . We consider the unitization of $\mathcal{L}(\mathcal{E})$ as $\mathcal{L}_1(\mathcal{E}) := \left\{ \begin{bmatrix} u & x \\ y^* & a \end{bmatrix} \mid a \in \mathcal{A}, u \in \mathbb{K}_1(\mathcal{E}), x, y \in \mathcal{E} \right\}$, where $\mathbb{K}_1(\mathcal{E}) = \mathbb{K}(\mathcal{E}) + \mathbb{C}I_{\mathcal{E}}$, ($I_{\mathcal{E}}$ is the identity operator on \mathcal{E} and when there is no confusion, it is denoted by I , for convenience we denote λI by λ for every complex scalar λ). The smallest operator subsystem of $\mathcal{L}_1(\mathcal{E})$ which contains \mathcal{A} and \mathcal{E} is denoted by $S_{\mathcal{A}}(\mathcal{E})$ and is defined as follow

$$S_{\mathcal{A}}(\mathcal{E}) := \begin{bmatrix} \mathbb{C}I & \mathcal{E} \\ \mathcal{E}^* & \mathcal{A} \end{bmatrix} = \left\{ \begin{bmatrix} \lambda & x \\ y^* & a \end{bmatrix} \mid a \in \mathcal{A}, \lambda \in \mathbb{C}, x, y \in \mathcal{E} \right\}.$$

For every natural number n , $\mathbb{M}_n(\mathcal{E})$ with its natural vector space structures and the following module action and inner product is a Hilbert C^* -module over the C^* -algebra $\mathbb{M}_n(\mathcal{A})$,

- (i) $(x_{ij}) \cdot (a_{ij}) := (\sum_{k=1}^n x_{ik} a_{kj})$ for every $(a_{ij}) \in \mathbb{M}_n(\mathcal{A})$ and $(x_{ij}) \in \mathbb{M}_n(\mathcal{E})$,
- (ii) $\langle (x_{ij}), (y_{ij}) \rangle := (\sum_{k=1}^n \langle x_{ki}, y_{kj} \rangle)$ for every $(x_{ij}), (y_{ij}) \in \mathbb{M}_n(\mathcal{E})$.

For Hilbert spaces H, K , and arbitrary given maps $\rho : \mathcal{A} \rightarrow \mathcal{B}(H)$, $\sigma : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(K)$ and $\Psi : \mathcal{E} \rightarrow \mathcal{B}(H, K)$, the map $\begin{bmatrix} u & x \\ y^* & a \end{bmatrix} \mapsto \begin{bmatrix} \sigma(u) & \Psi(x) \\ \Psi(y)^* & \rho(a) \end{bmatrix}$ from $\mathcal{L}_1(\mathcal{E})$ into $\mathcal{B}(K \oplus H)$ is denoted by $\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}$. Similarly, if $T \in \mathcal{B}(K)$, the map $\begin{bmatrix} \lambda & x \\ y^* & a \end{bmatrix} \mapsto \begin{bmatrix} \lambda T & \Psi(x) \\ \Psi(y)^* & \rho(a) \end{bmatrix}$ from $S_{\mathcal{A}}(\mathcal{E})$ into $\mathcal{B}(K \oplus H)$ is denoted by $\begin{bmatrix} T & \Psi \\ \Psi^* & \rho \end{bmatrix}$.

Assume that \mathcal{E}, \mathcal{F} are Hilbert C^* -modules over C^* -algebras \mathcal{A}, \mathcal{B} respectively, $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a completely positive map and $\Phi : \mathcal{E} \rightarrow \mathcal{F}$ is a linear map, we say

- (1) Φ is a φ -map, if $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$, for all $x, y \in \mathcal{E}$.
- (2) Φ is a semi- φ -map, if $\langle \Phi(x), \Phi(x) \rangle \leq \varphi(\langle x, x \rangle)$, for all $x \in \mathcal{E}$.
- (3) Φ is a completely semi- φ -map, if $\langle \Phi_n(x), \Phi_n(x) \rangle \leq \varphi_n(\langle x, x \rangle)$, for all $x \in \mathbb{M}_n(\mathcal{E})$ and $n \in \mathbb{N}$. If $\mathcal{B} = \mathcal{B}(H)$ and $\mathcal{F} = \mathcal{B}(H, K)$ for some Hilbert spaces H, K ,
- (4) Φ is non-degenerate, if $[\Phi(\mathcal{E})H] = K$.
- (5) Φ is a φ -representation or representation, if Φ is a φ -map and φ is a $*$ -representation.

(6) Φ is dilatable, if there exists a representation $\Psi : \mathcal{E} \rightarrow \mathcal{B}(H', K')$ and bounded operators $V : H \rightarrow H'$ and $W : K \rightarrow K'$ such that

$$\Phi(x) = W^* \Psi(x) V.$$

Remark 2.1. If $\rho : \mathcal{A} \rightarrow \mathcal{B}(H)$ is a $*$ -representation and $\Psi : \mathcal{E} \rightarrow \mathcal{B}(H, K)$ is a ρ -representation, then there exists a $*$ -representation $\sigma : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(K)$ such that for every $x, y \in \mathcal{E}$, $\sigma(x \otimes y) = \Phi(x) \Phi(y)^*$. Consequently, $\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(K \oplus H)$ is a representation of $\mathcal{L}_1(\mathcal{E})$. Conversely, every representation of the linking algebra has this form [1, Proposition 3.1].

Note that we use a weaker definition for non-degenerate operator valued maps on Hilbert C^* -modules rather than L. Arambašić's definition [1, Definition 3.2] for non-degenerate representations

on Hilbert C^* -modules. However, in the case of full Hilbert C^* -modules, nondegeneracy of the ρ -representation Ψ implies that $[\Psi(\mathcal{E})^*K] = H$, and consequently the two definitions coincide and σ , ρ and also $\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}$ are non-degenerate if and only if Ψ is non-degenerate by [1, Lemma 3.4].

Remark 2.2. Note that every φ -map is a completely semi- φ -map. Also if we consider \mathcal{A} as a Hilbert \mathcal{A} -module, then every unital completely positive map $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H)$ is a completely semi- φ -map on \mathcal{A} . In this case φ is a φ -map iff φ is a $*$ -representation. Thus, for every non-multiplicative unital completely positive map φ , there is a completely semi- φ -map which is not a φ -map. At the end of section 4, we provide a characterization of operator valued completely semi- φ -maps which helps us to construct completely semi- φ -maps which are not φ -map.

Proposition 2.3. Assume $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H)$ is a pure, unital completely positive map which is not multiplicative. Then φ is a completely semi- φ -map but it is not τ -map for any completely positive map $\tau : \mathcal{A} \rightarrow \mathcal{B}(H)$.

PROOF. Let (ρ, K, V) be the minimal Stinespring dilation triple for φ . Then $\rho : \mathcal{A} \rightarrow \mathcal{B}(K)$ is an irreducible representation for \mathcal{A} [2, Corollary 1.4.3]. If there exists a completely positive map $\tau : \mathcal{A} \rightarrow \mathcal{B}(H)$ such that $\varphi(x)^*\varphi(y) = \tau(a^*b)$ satisfies for every $x, y \in \mathcal{E}$, then $\tau \leq \varphi$, since φ is a completely semi- φ -map. Thus there exists a positive contraction $T \in \rho(\mathcal{A})'$ such that $\tau(a) = V^*T\rho(a)V$ for every $a \in \mathcal{A}$ [2, Theorem 1.4.2], but ρ is an irreducible representation, thus $\rho(\mathcal{A})' = \mathbb{C}.I_K$, so, $T = t.I_K$ for some scalar $t \in [0, 1]$. Therefore $\varphi(a)^*\varphi(b) = t\varphi(a^*b)$ for every $a, b \in \mathcal{E}$, which implies $t = 1$ and therefore φ is a φ -map, thus φ is multiplicative, which is a contradiction. \square

3. COMPLETELY SEMI- φ -MAPS, CP-EXTENDABILITY AND DILATABILITY

In the following, we show that each completely semi- φ -map Φ on a Hilbert C^* -module implements a completely positive map on the linking C^* -algebra. In fact, we show that φ and Φ are corners of a completely positive map on $\mathcal{L}_1(\mathcal{E})$. The following lemma can be obtained by [8, Lemma 3.1].

Lemma 3.1. Let \mathcal{A} be a unital C^* -algebra and \mathcal{E} a right Hilbert module over \mathcal{A} . Then for every $x \in \mathcal{E}$ and $a \in \mathcal{A}$, $\begin{bmatrix} 1 & x \\ x^* & a \end{bmatrix}$ is positive if and only if $\langle x, x \rangle_{\mathcal{A}} \leq a$.

Lemma 3.2. Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ be a completely positive map and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ be a linear map. Then Φ is a completely semi- φ -map if and only if $\begin{bmatrix} id & \Phi \\ \Phi^* & \varphi \end{bmatrix} : S_{\mathcal{A}}(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$ is a completely positive map

PROOF. Let Φ be a completely semi- φ -map and $\begin{bmatrix} 1 & x \\ x^* & a \end{bmatrix}$ a positive element of $S_{\mathcal{A}}(\mathcal{E})$. By above lemma, $\langle x, x \rangle_{\mathcal{A}} \leq a$ and so $\Phi(x)^*\Phi(x) \leq \varphi(\langle x, x \rangle_{\mathcal{A}}) \leq \varphi(a)$. Then, $\begin{bmatrix} 1 & \Phi(x) \\ \Phi(x)^* & \varphi(a) \end{bmatrix}$ is a positive element of $\mathcal{B}(H_2 \oplus H_1)$ and hence $\begin{bmatrix} id & \Phi \\ \Phi^* & \varphi \end{bmatrix}$ is a positive mapping. To show that $\begin{bmatrix} id & \Phi \\ \Phi^* & \varphi \end{bmatrix}$ is a completely positive map, let $(\begin{bmatrix} \lambda_{i,j} & T_{i,j} \\ S_{i,j}^* & a_{i,j} \end{bmatrix})_{i,j=1}^n$ be a positive element of $\mathbb{M}_n(S_{\mathcal{A}}(\mathcal{E}))$ for some $n \in \mathbb{N}$. By a unitary equivalence we have

$$(\begin{bmatrix} \lambda_{i,j} & T_{i,j} \\ S_{i,j}^* & a_{i,j} \end{bmatrix})_{i,j=1}^n \cong (\begin{bmatrix} (\lambda_{i,j})_{i,j}^n & (T_{i,j})_{i,j}^n \\ (S_{i,j}^*)_{i,j}^n & (a_{i,j})_{i,j}^n \end{bmatrix}) \in \mathbb{M}_2(\mathcal{L}_1(\mathbb{M}_n(\mathcal{E}))) \quad (1).$$

Then $(\lambda_{i,j})_{i,j}^n$ and $(a_{i,j})_{i,j}^n$ are positive matrices. First, we assume that $\lambda := (\lambda_{i,j})_{i,j}^n \in \mathbb{M}_n(\mathbb{C})$ is an invertible matrix. Set $T = (T_{i,j})_{i,j}^n \in \mathbb{M}_n(\mathcal{E})$ and $a = (a_{i,j})_{i,j}^n \in \mathbb{M}_n(\mathcal{A})$, then

$$\begin{bmatrix} I_n & \lambda^{-\frac{1}{2}}T \\ T^*\lambda^{-\frac{1}{2}} & a \end{bmatrix} = \begin{bmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \lambda & T \\ T^* & a \end{bmatrix} \begin{bmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & I_n \end{bmatrix}$$

is a positive element of $\mathbb{M}_2(\mathcal{L}_1(\mathbb{M}_n(\mathcal{E})))$. Thus $\langle \lambda^{-\frac{1}{2}}T, \lambda^{-\frac{1}{2}}T \rangle_{\mathbb{M}_n(\mathcal{A})} \leq a$ and hence

$$\Phi_n(\lambda^{-\frac{1}{2}}T)^* \Phi_n(\lambda^{-\frac{1}{2}}T) \leq \varphi_n(\langle \lambda^{-\frac{1}{2}}T, \lambda^{-\frac{1}{2}}T \rangle_{\mathbb{M}_n(\mathcal{A})}) \leq \varphi_n(a).$$

Then

$$\begin{bmatrix} I_n & \lambda^{-\frac{1}{2}}\Phi_n(T) \\ \Phi_n(T)^* \lambda^{-\frac{1}{2}} & \varphi_n(a) \end{bmatrix} = \begin{bmatrix} I_n & \Phi_n(\lambda^{-\frac{1}{2}}T) \\ \Phi_n(\lambda^{-\frac{1}{2}}T)^* & \varphi_n(a) \end{bmatrix}$$

is positive. Therefore

$$\left(\begin{bmatrix} \lambda_{i,j} & \Phi(T_{i,j}) \\ \Phi(T_{j,i})^* & \varphi(a_{i,j}) \end{bmatrix} \right)_{i,j} \cong \begin{bmatrix} \lambda & \Phi_n(T) \\ \Phi_n(T)^* & \varphi_n(a) \end{bmatrix} = \begin{bmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & \lambda^{-\frac{1}{2}}\Phi_n(T) \\ \Phi_n(T)^* \lambda^{-\frac{1}{2}} & \varphi_n(a) \end{bmatrix} \begin{bmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & I_n \end{bmatrix}$$

is positive too. This means that $\begin{bmatrix} id & \Phi \\ \Phi^* & \varphi \end{bmatrix}$ is a completely positive map. In general case, if λ is not invertible we can use $\lambda + rI_n$ for some $r > 0$.

Conversely, assume $\begin{bmatrix} id & \Phi \\ \Phi^* & \varphi \end{bmatrix}$ is a completely positive map on $S_{\mathcal{A}}(\mathcal{E})$. Since for every $x \in \mathbb{M}_n(\mathcal{E})$, $\begin{bmatrix} 1 & x \\ x^* & \langle x, x \rangle \end{bmatrix}$ is a positive element of $S_{\mathbb{M}_n(\mathcal{A})}(\mathbb{M}_n(\mathcal{E}))$, $\begin{bmatrix} 1 & \Phi_n(x) \\ \Phi_n(x)^* & \varphi_n(\langle x, x \rangle) \end{bmatrix}$ is positive. Therefore Φ is a completely semi- φ -map. \square

Theorem 3.3. *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ be a completely positive map and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ a completely semi- φ -map. Then there exists a unital completely positive map $\psi : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(H_2)$ such that $\begin{bmatrix} \psi & \Phi \\ \Phi^* & \varphi \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$ is a completely positive map.*

PROOF. By the above lemma, $\theta_0 = \begin{bmatrix} id & \Phi \\ \Phi^* & \phi \end{bmatrix} : S_{\mathcal{A}}(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$ is a completely positive map. Without loss of generality we can assume that θ_0 is unital, since for every positive real number r , the map $\begin{bmatrix} id & r\Phi \\ r\Phi^* & r^2\varphi \end{bmatrix}$ is completely positive. By Arveson's extension theorem, θ_0 has a unital completely positive extension $\theta : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$. Put $p := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. By [5, Corollary 5.2.2], for each $u \in \mathbb{K}_1(\mathcal{E})$ we have

$$\theta \left(\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \right) = \theta(p \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} p) = \theta(p) \theta \left(\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \right) \theta(p) = \begin{bmatrix} id_{H_2} & 0 \\ 0 & 0 \end{bmatrix} \theta \left(\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} id_{H_2} & 0 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{B}(H_2) & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, θ is a corner preserving unital completely positive map on $\mathcal{L}_1(\mathcal{E})$. Therefore, $\psi := \theta|_{\mathbb{K}_1(\mathcal{E})}$ is a unital completely positive map from $\mathbb{K}_1(\mathcal{E})$ into $\mathcal{B}(H_2)$ such that $\theta = \begin{bmatrix} \psi & \Phi \\ \Phi^* & \varphi \end{bmatrix}$. \square

The next theorem is a strengthened form of the main theorem of [3].

Theorem 3.4. *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ be a completely positive map and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ be a completely semi- φ -map. Then there exist Hilbert spaces K_1, K_2 , a bounded operator $V : H_1 \rightarrow K_1$, an isometry $W : H_2 \rightarrow K_2$, a $*$ -homomorphism $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$ and a ρ -representation $\Psi : \mathcal{E} \rightarrow \mathcal{B}(K_1, K_2)$ such that for all $a \in \mathcal{A}$ and $x \in \mathcal{E}$,*

$$\varphi(a) = V^* \rho(a) V \quad \Phi(x) = W^* \Psi(x) V.$$

Furthermore, if φ is unital, then V is an isometry.

PROOF. By the previous theorem, $\begin{bmatrix} id & \Phi \\ \Phi^* & \varphi \end{bmatrix} : S_{\mathcal{A}}(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$ has a completely positive map extension $\theta = \begin{bmatrix} \psi & \Phi \\ \Phi^* & \varphi \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$.

By Stinespring's theorem for a completely positive maps on C^* -algebras there is a triple (π, K, W) consists of a unital $*$ -representation $\pi : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(K)$ and an operator $W \in \mathcal{B}(H_2 \oplus H_1, K)$ such that for every $X \in \mathcal{L}_1(\mathcal{E})$ the following holds

$$\theta(X) = W^* \pi(X) W.$$

Similar to [1, Proposition 3.1], we set $K_1 = [\pi(\begin{bmatrix} 0 & 0 \\ 0 & \mathcal{A} \end{bmatrix})K]$ and $K_2 = [\pi(\begin{bmatrix} \mathbb{K}_1(\mathcal{E}) & 0 \\ 0 & 0 \end{bmatrix})K]$. Hence $K \cong K_2 \oplus K_1$ and we can write $\pi = \begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}$ and $W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \in \mathcal{B}(H_2 \oplus H_1, K_2 \oplus K_1)$, where $\sigma : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(K_2)$ and $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$ are $*$ -representations and $\Psi : \mathcal{E} \rightarrow \mathcal{B}(K_1, K_2)$ is a σ - ρ -representation.

Then for every $\begin{bmatrix} T & x \\ y^* & a \end{bmatrix} \in \mathcal{L}_1(\mathcal{E})$ we have

$$\begin{bmatrix} \psi(T) & \Phi(x) \\ \Phi(y)^* & \varphi(a) \end{bmatrix} = \begin{bmatrix} W_1^* & W_3^* \\ W_2^* & W_4^* \end{bmatrix} \begin{bmatrix} \sigma(T) & \Psi(x) \\ \Psi(y)^* & \rho(a) \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \quad (2).$$

In the above equation, set $T = I_{\mathcal{E}}$, $x = y = 0$ and $a = 0$. Since σ and ρ are unital maps, one has

$$\begin{bmatrix} id_{H_2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} W_1^* & W_3^* \\ W_2^* & W_4^* \end{bmatrix} \begin{bmatrix} id_{K_2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix},$$

thus $W_1^* W_1 = id_{H_2}$ and $W_2^* W_2 = 0$. Now, set $T = 0$, $y = 0$, $a = 0$ and an arbitrary $x \in \mathcal{E}$ in equation (2), easy calculation shows that $\Phi(x) = W_1^* \Psi(x) W_4$. Finally, setting $T = 0$, $x = y = 0$ and an arbitrary element $a \in A$ in equation (2) shows that $\varphi(a) = W_4^* \rho(a) W_4$ and $W_3^* \rho(a) W_3 = 0$. Since ρ is unital, $W_3 = 0$. If φ is unital, one has

$$id_{H_1} = \varphi(1) = W_4^* \rho(1) W_4 = W_4^* id_{K_1} W_4 = W_4^* W_4.$$

□

We summarize the results of this section on completely semi- φ -maps in the following corollary:

Corollary 3.5. *Let \mathcal{A} be a unital C^* -algebra and \mathcal{E} a right Hilbert \mathcal{A} -module. For every pair of given maps $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ the following are equivalent:*

(i) Φ is a completely semi- φ -map

(ii) $\begin{bmatrix} id & \Phi \\ \Phi^* & \varphi \end{bmatrix} : S_{\mathcal{A}}(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$ is a completely positive map

(iii) There exists a unital completely positive map $\psi : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(H_2)$ such that $\begin{bmatrix} \psi & \Phi \\ \Phi^* & \varphi \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$ is a completely positive map

(v) There exist Hilbert spaces K_1, K_2 , a bounded operator $V : H_1 \rightarrow K_1$, an isometry $W : H_2 \rightarrow K_2$, and a unital $*$ -representation $\pi : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(K_2 \oplus K_1)$ such that

$$\begin{bmatrix} * & \Phi \\ \Phi^* & \varphi \end{bmatrix} (\cdot) = \begin{bmatrix} W^* & 0 \\ 0 & V^* \end{bmatrix} \pi(\cdot) \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix},$$

(iv) There exists a pair $((\rho, K_1, V), (\Psi, K_2, W))$ consists of Hilbert spaces K_1, K_2 , a bounded operator $V : H_1 \rightarrow K_1$, an isometry $W : H_2 \rightarrow K_2$, a unital $*$ -representation $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$, and a ρ -representation $\Psi : \mathcal{E} \rightarrow \mathcal{B}(K_1, K_2)$ such that

$$\varphi(a) = V^* \rho(a) V, \quad \Phi(x) = W^* \Psi(x) V,$$

for all $a \in A$ and $x \in \mathcal{E}$.

PROOF. (i) \Leftrightarrow (ii) by Lemma 3.2. (v) \Leftrightarrow (iv) and (v) \Rightarrow (iii) by Remark 2.1. (i) \Rightarrow (iv) by Theorem 3.4 and obviously (iii) \Rightarrow (ii). □

4. UNIQUENESS OF MINIMAL DILATION PAIRS

Assume \mathcal{A} is a unital C^* -algebra and \mathcal{E} is a right Hilbert \mathcal{A} -module. As it is shown in the previous section, if $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H)$ is a completely positive map, every completely semi- φ -map on \mathcal{E} is dilatable. In this section, we show that every completely semi- φ -map on a Hilbert C^* -module has a minimal dilation pair. Furthermore, we show that two minimal dilation pairs for a given completely semi- φ -map are unitarily equivalent and implement unitarily equivalent $*$ -representations on the linking C^* -algebra of \mathcal{E} .

Definition 4.1. *Let \mathcal{A} be a C^* -algebra and \mathcal{E} a right Hilbert \mathcal{A} -module. A map $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H, K)$ is a CP-extendable map, if there exist completely positive maps $\psi : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(K)$ and $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H)$ such that $\begin{bmatrix} \psi & \Phi \\ \Phi^* & \varphi \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(K \oplus H)$ is a completely positive map. In this case we call the pair (φ, Φ) a CP-extendable pair.*

As it is shown in Corollary 3.5, if $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H)$ is a completely positive map, then, every completely semi- φ -map $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H, K)$ has a CP-extension on the linking C^* -algebra $\mathcal{L}(\mathcal{E})$ which acts block-wise. Thus Φ is a CP-extendable map and (φ, Φ) is a CP-extendable pair.

Since completely semi- φ -maps are CP-extendable, the following theorem is a generalization of Corollary 3.5 and can be proved, by using Stinespring's theorem for linking C^* -algebra and [1, Proposition 3.1].

Theorem 4.2. *Let \mathcal{A} be a unital C^* -algebra and \mathcal{E} a right Hilbert \mathcal{A} -module. For a given map $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ the following are equivalent:*

- (i) Φ is CP-extendable,
- (ii) *There exist Hilbert spaces K_1, K_2 , bounded operators $V : H_1 \rightarrow K_1$, $W : H_2 \rightarrow K_2$ and a unital $*$ -representation $\pi : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(K_2 \oplus K_1)$ and a completely positive map $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ such that*

$$\begin{bmatrix} * & \Phi \\ \Phi^* & \varphi \end{bmatrix} (\cdot) = \begin{bmatrix} W^* & 0 \\ 0 & V^* \end{bmatrix} \pi(\cdot) \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix},$$

- (iii) Φ is dilatable.

In the following we recall a definition from [4] and show that [4, Theorem 2.4] holds for completely semi- φ -maps.

Definition 4.3. *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ be a completely positive map and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ be a completely semi- φ -map. A dilation pair for (φ, Φ) is a pair of triples $((\rho, K_1, V), (\Psi, K_2, W))$ consists of Hilbert spaces K_1, K_2 , a unital $*$ -representation $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$ and a ρ -morphism $\Psi : \mathcal{E} \rightarrow \mathcal{B}(K_1, K_2)$ and bounded operators $V : H_1 \rightarrow K_1$ and $W : H_2 \rightarrow K_2$, such that*

$$\Phi(x) = W^* \Psi(x) V \quad , \quad \varphi(a) = V^* \rho(a) V,$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{E}$. A dilation pair $((\rho, K_1, V), (\Psi, K_2, W))$ is called minimal when the following conditions are satisfied

- (i) $[\rho(\mathcal{A})VH_1] = K_1$,
- (ii) Ψ be a nondegenerate map.

Suppose that $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ is a completely positive map and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ is a completely semi- φ -map. By Theorem 3.4 there exists a dilation pair $((\rho, K_1, V), (\Psi, K_2, W))$ for (φ, Φ) , such that W is an isometry. We can replace (ρ, K_1, V) by a minimal Stinespring dilation triple for φ . So without loss of generality we can assume that (ρ, K_1, V) is a minimal Stinespring dilation triple for φ . Set $\mathcal{L} := [\Psi(\mathcal{E})VH_1] = [\Psi(\mathcal{E})K_1]$ and define $\Gamma : \mathcal{E} \rightarrow \mathcal{B}(K_1, L)$ by

$$\Gamma(x)k := \Psi(x)k$$

for all $x \in \mathcal{E}$ and $k \in K_1$.

Assume $j : L \rightarrow K_2$ is the inclusion map of L into K_2 , so, j^* is the orthogonal projection from K_2 onto L . Thus $\Gamma(x) = j^*\Psi(x)$ for every $x \in \mathcal{E}$. Note that Ψ is a ρ -morphism, therefore Ψ is a ρ -module map. Thus for every $a \in \mathcal{A}$, $h \in H_1$ and $x, y \in \mathcal{E}$

$$\Gamma(x)^*\Gamma(y)(\rho(a)Vh) = \rho(\langle x, y \rangle)(\rho(a)Vh).$$

Since $[\rho(A)VH_1] = K_1$, Γ is a ρ -map. Now define $T : L \rightarrow H_2$ by $T(l) := W^*(l)$ for all $l \in L$. Consider $S := T^* \in \mathcal{B}(H_2, L)$, then $((\rho, K_1, V), (\Gamma, L, S))$ is a minimal dilation pair for (φ, Φ) . Note that W is an isometry and $T = W^*j$, thus T and $S = T^*$ are contractions with norm one.

The following theorem on the uniqueness of minimal dilation pairs of completely semi- φ -maps is in fact the same as [4, Theorem 2.4].

Theorem 4.4. *Let Φ and φ be as in definition 4.3. Assume $((\rho, K_1, V), (\Psi, K_2, W))$ and $((\pi, L_1, U), (\Gamma, L_2, S))$ are two minimal dilation pairs for (φ, Φ) . Then there exist unitary operators $T_1 : K_1 \rightarrow L_1$ and $T_2 : K_2 \rightarrow L_2$ such that*

- (i) $T_1V = U$ and $T_1\rho(a) = \pi(a)T_1$ for all $a \in \mathcal{A}$.
- (ii) $T_2W = S$ and $T_2\Psi(x) = \Gamma(x)T_1$ for all $x \in \mathcal{E}$.
- (iii) $\begin{bmatrix} T_2 & 0 \\ 0 & T_1 \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & U \end{bmatrix}$ and $\begin{bmatrix} T_2 & 0 \\ 0 & T_1 \end{bmatrix} \begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix} \begin{bmatrix} T_2^* & 0 \\ 0 & T_1^* \end{bmatrix} = \begin{bmatrix} \tau & \Gamma \\ \Gamma^* & \pi \end{bmatrix}$, where $\sigma : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(K_2)$ and $\tau : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(L_2)$ are unique $*$ -homomorphisms which satisfy the equations $\sigma(x \otimes y) = \Psi(x)\Psi(y)^*$ and $\tau(x \otimes y) = \Gamma(x)\Gamma(y)^*$, for all $x, y \in \mathcal{E}$.

Consequently, representations ρ and $\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}$ and σ are unitarily equivalent to representations π and $\begin{bmatrix} \tau & \Gamma \\ \Gamma^* & \pi \end{bmatrix}$ and τ , respectively.

PROOF. (i) and (ii) have the same proof as [4, Theorem 2.4] and (iii) can be obtained from (i) and (ii). \square

Remark 4.5. *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ be a completely positive map and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ be a completely semi- φ -map. By the preceding discussion on the existence of a minimal dilation pair for completely semi- φ -maps, there is a minimal dilation pair $((\pi, L_1, U), (\Gamma, L_2, S))$ for (φ, Φ) such that S is contractive. Theorem 4.4 implies that for every minimal dilation pair $((\pi', L'_1, U'), (\Gamma', L'_2, S'))$ for (φ, Φ) , S' is contractive. On the other hand by [4, Theorem 2.1] for every φ -map there is a minimal dilation pair $((\rho, K_1, V), (\Psi, K_2, W))$ such that W is coisometry, thus, by Theorem 4.4 if $((\rho', K'_1, V'), (\Psi', K'_2, W'))$ is another minimal dilation pair for the φ -map, then W' is coisometry. Therefore there exist many examples of completely semi- φ -maps which are not φ -map.*

In the following we show that this new notion of dilation for completely semi- φ -maps is compatible with the previous notion of minimal dilation pair for completely positive maps on C^* -algebras. For this purpose we recall the definition of irreducible maps on Hilbert C^* -modules and show that a unital completely positive map on a C^* -algebra is pure if and only if its minimal dilation pair (in sense of Definition 4.3) is irreducible.

Definition 4.6. *Let $\Psi : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ be a map and $K_1 \leq H_1$ and $K_2 \leq H_2$. The pair (K_1, K_2) is said to be Ψ -invariant if $\Psi(\mathcal{E})K_1 \subseteq K_2$ and $\Psi(\mathcal{E})^*K_2 \subseteq K_1$. Ψ is said to be irreducible if $(0, 0)$ and (H_1, H_2) are the only Ψ -invariant pairs.*

Remark 4.7. *The above definition is a modification of Definition 3.3 [1], just we state it for every map not just representations. Arambašić showed that if $\rho : \mathcal{A} \rightarrow \mathcal{B}(H)$ is a $*$ -representation and $\Psi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a ρ -representation for \mathcal{E} such that $[\Psi(\mathcal{E})H] = K$ then Ψ is irreducible (in sense of Definition 4.6) if and only if ρ is irreducible [1, Proposition 3.6].*

By a result of Arveson [2, Corollary 1.4.3] the completely positive map φ is pure if and only if it can be dilated to an irreducible $*$ -representation of \mathcal{A} (in other words, its minimal Stinespring dilation triple is irreducible). The following corollary is a generalization of this fact.

Corollary 4.8. *Let \mathcal{A} be a unital C^* -algebra and $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H)$ be a unital completely positive map. Assume $((\rho, K_1, V), (\Psi, K_2, W))$ is the minimal dilation pair for (φ, φ) . Then φ is pure if and only if Ψ is an irreducible map.*

PROOF. Assume φ is pure, thus by [2, Corollary 1.4.3] every minimal Stinespring dilation triple of it is irreducible. Since $((\rho, K_1, V), (\Psi, K_2, W))$ is a minimal dilation pair for (φ, φ) , ρ is an irreducible $*$ -representation and $[\Psi(\mathcal{E})K_1] = K_2$, therefore Ψ is an irreducible representation of \mathcal{E} by [1, Proposition 3.6]. Conversely, If $((\rho, K_1, V), (\Psi, K_2, W))$ is a dilation pair for (φ, φ) such that Ψ is irreducible, then [1, Lemma 3.5] implies that ρ is an irreducible $*$ -representation for \mathcal{A} . Thus (ρ, K_1, V) is a minimal Stinespring dilation triple for φ , and $((\rho, K_1, V), (\Psi, K_2, W))$ is a minimal dilation pair for (φ, φ) . Therefore φ is pure by [2, Corollary 1.4.3]. \square

5. A RADON-NIKODYM-TYPE THEOREM FOR COMPLETELY SEMI- φ -MAPS

We denote the set of all pairs (φ, Φ) , where $\varphi : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ is a completely positive map and $\Phi : \mathcal{F} \rightarrow \mathcal{B}(H_1, H_2)$ is a completely semi- φ -map, by $\mathcal{CP}\mathcal{E}(\mathcal{F}, H_1, H_2)$.

Let $(\varphi, \Phi) \in \mathcal{CP}\mathcal{E}(\mathcal{F}, H_1, H_2)$. Assume $((\rho, K_1, V), (\Psi, K_2, W))$ is a minimal dilation pair for (φ, Φ) . Then ρ is unital and by Remark 2.1 there exists a $*$ -homomorphism $\sigma : \mathbb{K}_1(\mathcal{F}) \rightarrow \mathcal{B}(K_2)$ such that $\sigma(x \otimes y) = \Psi(x)\Psi(y)^*$ and moreover

$$\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix} : \mathcal{L}_1(\mathcal{F}) \rightarrow \mathcal{B}(K_2 \oplus K_1), \quad \begin{bmatrix} u & x \\ y^* & a \end{bmatrix} \mapsto \begin{bmatrix} \sigma(u) & \Psi(x) \\ \Psi(y)^* & \rho(a) \end{bmatrix}.$$

is a $*$ -representation.

The range of $\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}$ is a C^* -subalgebra of $\mathcal{B}(K_2 \oplus K_1)$ and it is easy to check that its commutant is the set of all $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \in \mathcal{B}(K_2 \oplus K_1)$ such that

$$P\Psi(x) = \Psi(x)Q, \quad Q\Psi(x)^* = \Psi(x)^*P, \quad (1)$$

$$\sigma(u)P = P\sigma(u), \quad \rho(a)Q = Q\rho(a) \quad (2)$$

for all $x \in \mathcal{F}$, $a \in \mathcal{A}$ and $u \in \mathbb{K}_1(\mathcal{F})$.

If \mathcal{F} is full, then (1) implies (2). The above discussion lead us to the following definition [1, Definition 4.1].

Definition 5.1. *Let $\rho : \mathcal{A} \rightarrow \mathcal{B}(H_1)$ be a unital $*$ -representation and $\Psi : \mathcal{F} \rightarrow \mathcal{B}(H_1, H_2)$ a ρ -map. Commutant of Ψ is the set of all operators $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \in \mathcal{B}(H_2 \oplus H_1)$ such that the following equations hold for all $x \in \mathcal{F}$*

$$P\Psi(x) = \Psi(x)Q, \quad Q\Psi(x)^* = \Psi(x)^*P,$$

and is denoted by $\Psi(\mathcal{F})'$.

Remark 5.2. *Assume Ψ and ρ as in the Definition 5.1. Then $\Psi(\mathcal{F})'$ is a C^* -algebra, moreover $(\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}(\mathcal{L}(\mathcal{F})))' \subseteq \Psi(\mathcal{F})'$. In the case of full Hilbert C^* -modules if Ψ is non-degenerate ($[\Psi(\mathcal{F})H_1] = H_2$), then $[\Psi(\mathcal{F})^*H_2] = H_1$ and $(\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}(\mathcal{L}(\mathcal{F})))' = \Psi(\mathcal{E})'$ [1, Lemma 4.3 and Lemma 4.4].*

From now on we deal with full Hilbert C^* -modules. In the following we define an order relation on $\mathcal{CP}\mathcal{E}(\mathcal{F}, H, K)$ and prove a Radon-Nikodym type theorem for this class of maps.

Definition 5.3. *Let $(\varphi_i, \Phi_i) \in \mathcal{CP}\mathcal{E}(\mathcal{F}, H, K)$ for $i = 1, 2$. We say that $(\varphi_1, \Phi_1) \ll (\varphi_2, \Phi_2)$ when*

$$\begin{bmatrix} id & \Phi_1 \\ \Phi_1^* & \varphi_1 \end{bmatrix} \leq_{cp} \begin{bmatrix} id & \Phi_2 \\ \Phi_2^* & \varphi_2 \end{bmatrix},$$

where \leq_{cp} is the order on the set of completely positive maps from $S_{\mathcal{A}}(\mathcal{F})$ into $\mathcal{B}(K \oplus H)$.

We use the notation $T \oplus S$ instead of $\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$ for operators $T \in \mathcal{B}(H)$ and $S \in \mathcal{B}(K)$. Note that $T \oplus S$ is a positive operator on $H \oplus K$ if and only if $T \in \mathcal{B}(H)_+$ and $S \in \mathcal{B}(K)_+$. The following proposition is similar to [7, Lemma 2.10] on φ -maps, and we show that the lemma is true for completely semi- φ -maps, too.

Proposition 5.4. *Assume $(\varphi, \Phi) \in \mathcal{CP}\mathcal{E}(\mathcal{E}, H_1, H_2)$ and $((\rho, K_1, V), (\Psi, K_2, W))$ is a minimal dilation pair of (φ, Φ) . For every positive operator $T \oplus S \in \Psi(\mathcal{E})'$ define the map $\Phi_{T \oplus S} : \mathcal{E} \rightarrow \mathcal{B}(H_1, H_2)$ by*

$$\Phi_{T \oplus S}(x) := W^* T^{\frac{1}{2}} \Psi(x) S^{\frac{1}{2}} V$$

for all $x \in \mathcal{E}$. Then $(\varphi_S, \Phi_{T \oplus S})$ is a CP-extendable pair, where $\varphi_S(a) = V^* S \rho(a) V$, for each $a \in \mathcal{A}$. Moreover, if T is contractive, $\Phi_{T \oplus S}$ is a completely semi- φ_S -map.

PROOF. Since $((\rho, K_1, V), (\Psi, K_2, W))$ is a minimal dilation for (φ, Φ) , there exists a non-degenerate (and therefore unital) $*$ -homomorphism $\sigma : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(K_2)$ such that

$$\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(K_2 \oplus K_1)$$

is a $*$ -homomorphism, so

$$\begin{bmatrix} W^* T^{\frac{1}{2}} & 0 \\ 0 & V^* S^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix} \begin{bmatrix} T^{\frac{1}{2}} W & 0 \\ 0 & S^{\frac{1}{2}} V \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$$

is a completely positive map. By the discussion previous the Definition 5.1, $T \in \sigma(\mathbb{K}_1(\mathcal{E}))'$ and $S \in \rho(\mathcal{A})'$ thus $S^{\frac{1}{2}} \in \rho(\mathcal{A})'$ and $T^{\frac{1}{2}} \in \sigma(\mathbb{K}_1(\mathcal{E}))'$, therefore

$$\begin{bmatrix} W^* T^{\frac{1}{2}} \sigma(\cdot) T^{\frac{1}{2}} W & W^* T^{\frac{1}{2}} \Psi(\cdot) S^{\frac{1}{2}} V \\ V^* S^{\frac{1}{2}} \Psi(\cdot)^* T^{\frac{1}{2}} W & V^* S \rho(\cdot) V \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(H_2 \oplus H_1)$$

is a completely positive map, thus $\Phi_{T \oplus S}$ is a CP-extendable map and $(\varphi_S, \Phi_{T \oplus S})$ is a CP-extendable pair.

It is easy to check that $\Phi_{T \oplus S}$ is a completely semi- φ_S -map when T is contractive. \square

The above proposition has a converse that is a Radon-Nikodym type theorem for completely semi- φ -maps.

Theorem 5.5. *Let \mathcal{E} be a full Hilbert C^* -module over a unital C^* -algebra \mathcal{A} . Assume $(\varphi_1, \Phi_1), (\varphi_2, \Phi_2) \in \mathcal{CP}\mathcal{E}(\mathcal{E}, H_1, H_2)$ and $(\varphi_1, \Phi_1) \ll (\varphi_2, \Phi_2)$. Then, there exists a dilation pair $((\rho, K_1, W'), (\Psi, K_2, W))$ for (φ_2, Φ_2) such that Ψ is non-degenerate and a unique positive contraction $T \oplus S \in \Psi(\mathcal{E})'$ such that*

$$\Phi_1(e) = W^* T^{\frac{1}{2}} \Psi(e) S^{\frac{1}{2}} W' \quad (3)$$

for all $a \in \mathcal{A}$ and $e \in \mathcal{E}$.

PROOF. Assume $(\varphi_1, \Phi_1) \ll (\varphi_2, \Phi_2)$, thus, $\begin{bmatrix} id & \Phi_1 \\ \Phi_1^* & \varphi_1 \end{bmatrix} \leq_{cp} \begin{bmatrix} id & \Phi_2 \\ \Phi_2^* & \varphi_2 \end{bmatrix}$. Put $\psi_1 := \begin{bmatrix} id & \Phi_1 \\ \Phi_1^* & \varphi_1 \end{bmatrix}$ and $\psi_2 := \begin{bmatrix} id & \Phi_2 \\ \Phi_2^* & \varphi_2 \end{bmatrix}$. Thus $\psi_2 - \psi_1$ is a completely positive map. By Arveson's extension theorem, $\widetilde{\psi_2 - \psi_1}$ and ψ_1 have completely positive extensions $\widetilde{\psi_2 - \psi_1}$ and $\widetilde{\psi_1}$ on $\mathcal{L}_1(\mathcal{E})$. Let $\widetilde{\psi_2} := \widetilde{\psi_1} + \widetilde{\psi_2 - \psi_1}$. We have $\widetilde{\psi_1} \leq_{cb} \widetilde{\psi_2}$. Thus $\widetilde{\psi_2}$ is a completely positive extension for ψ_2 such that $\widetilde{\psi_1} \leq_{cb} \widetilde{\psi_2}$. Assume (π, K, V) is the minimal Stinespring dilation triple for $\widetilde{\psi_2}$, then similar to the proof of Theorem 3.3, K decomposes to $K_2 \oplus K_1$ and there exist unital $*$ -homomorphisms $\sigma : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(K_2)$ and $\rho : \mathcal{A} \rightarrow \mathcal{B}(K_1)$ and also a σ - ρ -representation $\Psi : \mathcal{E} \rightarrow \mathcal{B}(K_1, K_2)$ such that $\pi = \begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}$ and $V = W \oplus W' \in \mathcal{B}(H_2 \oplus H_1, K_2 \oplus K_1)$. Therefore $((\rho, K_1, W'), (\Psi, K_2, W))$ is a dilation pair for (φ_2, Φ_2) and $\widetilde{\psi_1} \leq_{cp} \begin{bmatrix} W^* & 0 \\ o & W'^* \end{bmatrix} \begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & W' \end{bmatrix}$. Note that π is non-degenerate and by the assumption \mathcal{E} is full, therefore Ψ is non-degenerate and $\Psi(\mathcal{E})' = \pi(\mathcal{L}_1(\mathcal{E}))'$

by Remark 5.2 and Remark 2.1. Thus by [2, Theorem 1.4.2], there is a unique $T \oplus S \in \Psi(\mathcal{E})'$ such that $0 \leq T \oplus S \leq id_{K_2 \oplus K_1}$ and

$$\widetilde{\psi}_1 = \begin{bmatrix} W^* & 0 \\ 0 & W'^* \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & W' \end{bmatrix} = \begin{bmatrix} W^* T \sigma(\cdot) W & W^* T \Psi(\cdot) W' \\ W'^* S \Psi(\cdot)^* W & W'^* S \rho(\cdot) W' \end{bmatrix}.$$

Since $\widetilde{\psi}_1$ is an extension of $\begin{bmatrix} id & \Phi_1 \\ \Phi_1^* & \varphi_1 \end{bmatrix}$, one has $\Phi_1(x) = W^* T \Psi(x) W'$ and $\varphi_1(a) = W'^* S \rho(a) W'$ for all $x \in \mathcal{E}$ and $a \in \mathcal{A}$. But note that $\Psi(\mathcal{E})'$ is a C^* -algebra, hence $T^{\frac{1}{2}} \oplus S^{\frac{1}{2}} = (T \oplus S)^{\frac{1}{2}} \in \Psi(\mathcal{E})'$, thus $T \Psi(x) = T^{\frac{1}{2}} T^{\frac{1}{2}} \Psi(x) = T^{\frac{1}{2}} \Psi(x) S^{\frac{1}{2}}$. Then $\Phi_1(x) = W^* T^{\frac{1}{2}} \Psi(x) S^{\frac{1}{2}} W'$. □

Acknowledgment. The research of the first author was in part supported by a grant from IPM (No. 94470046).

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